

## ENLARGEABILITY, FOLIATIONS, AND POSITIVE SCALAR CURVATURE

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ABSTRACT. We extend the deep and important results of Lichnerowicz, Connes, and Gromov-Lawson which relate geometry and characteristic numbers to the existence and non-existence of metrics of positive scalar curvature. In particular, we show that a spin foliation with Hausdorff homotopy groupoid of an enlargeable manifold admits no metric of positive scalar curvature.

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## 1. INTRODUCTION

In this paper, we extend the famous results of Lichnerowicz, [L62], Connes, [C86], and Gromov and Lawson, [GL80a, GL80b, GL83] on the relationship of geometry and characteristic numbers to the existence and non-existence of metrics of positive scalar curvature. We assume that we have a spin foliation  $F$  with Hausdorff homotopy groupoid on a compact manifold  $M$ . The condition on the homotopy groupoid allows us to employ the index theory for foliations developed in [H95, HL99]. Our main result is that if  $M$  is enlargeable (a very large and important collection of manifolds), then  $F$  does not admit any metric of positive scalar curvature.

In [L62], Lichnerowicz proved that for any bundle of spinors  $\mathcal{S}$  over a spin manifold  $M$ , the Atiyah-Singer operator  $\not{D}$  and the connection Laplacian  $\nabla^*\nabla$  on  $\mathcal{S}$  are related by

$$\not{D}^2 = \nabla^*\nabla + \frac{1}{4}\kappa,$$

where  $\kappa$  is the scalar curvature of  $M$ , that is  $\kappa = -\sum_{i,j=1}^n \langle R_{e_i, e_j}(e_i), e_j \rangle$ , where  $e_1, \dots, e_n$  is any local orthonormal framing of the tangent bundle of  $M$  and  $R$  is the curvature operator on  $M$ . This leads immediately to the following.

**Theorem 1.1.** [L62] *If  $M$  is a compact spin manifold and  $\hat{A}(M) \neq 0$ , then  $M$  does not admit any metric of positive scalar curvature.*

This theorem and its generalizations have important and deep consequences. Some of the most far reaching were obtained by Connes and by Gromov and Lawson.

In the seminal paper [C86], Connes proved that for any transversely oriented foliated manifold, integration over the transverse fundamental class yields a well defined map from the K-theory of the canonically

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associated  $C^*$  algebra to  $\mathbb{C}$ . He derives many important consequences from this, including the following extension of Theorem 1.1.

**Theorem 1.2.** [C86] *If  $M$  is a compact oriented manifold with  $\hat{A}(M) \neq 0$ , then no spin foliation of  $M$  has a metric of positive scalar curvature.*

Note that  $\hat{A}(M)$  need not be an integer, since  $M$  is not assumed to be spin.

Recently Zhang, [Z16], has proven the following theorem, which shifts the spin assumption to  $M$ .

**Theorem 1.3.** [Z16] *If  $M$  is a compact oriented spin manifold with  $\hat{A}(M) \neq 0$ , then no foliation of  $M$  has a metric of positive scalar curvature.*

For the results of Lichnerowicz and Connes,  $\hat{A}(M)$  is the usual Hirzebruch  $\hat{A}$  genus which occurs in dimensions  $4k$ . The result of Zhang also includes the Atiyah-Milnor-Singer  $\mathbb{Z}_2 \hat{A}$  invariant, which occurs in dimensions  $8k + 1$  and  $8k + 2$ . See [LM89], II.7.

In [GL80a, GL80b, GL83], Gromov-Lawson introduced the notion of enlargeable manifolds. This is a vast and important set of compact manifolds, which includes all solvmanifolds, all manifolds which admit metrics of non-positive sectional curvature, all sufficiently large 3-manifolds, as well as many families of important  $K(\pi, 1)$ -manifolds. The category of enlargeable manifolds is closed under products, connected sums (with anything), and changes of differential structure.

Recall the following definitions from [GL83].

**Definition 1.4.** *A  $C^1$  map  $f : M \rightarrow M'$  between Riemannian manifolds is  $\epsilon$  contracting if  $\|f_*(v)\| \leq \epsilon \|v\|$  for all tangent vectors  $v$  to  $M$ .*

Denote by  $\mathbb{S}^n(1)$  the usual  $n$  sphere of radius 1.

**Definition 1.5.** *A compact Riemannian  $n$ -manifold is enlargeable if for every  $\epsilon > 0$ , there is a orientable Riemannian covering which admits an  $\epsilon$  contracting map onto  $\mathbb{S}^n(1)$  which is constant near infinity and has non-zero degree.*

Gromov-Lawson proved several important non-existence theorems, including the following.

**Theorem 1.6** ([GL83]). *An enlargeable spin manifold does not admit any metric of positive scalar curvature.*

In this paper, we extend the Gromov-Lawson result as follows.

**Theorem 1.7.** *If  $M$  is an enlargeable manifold, then no spin foliation of  $M$  with Hausdorff homotopy groupoid has a metric of positive scalar curvature.*

**Example 1.8.** *The most classical examples of enlargeable manifolds are tori. In [Z16], Zhang announced, but without the details of the proof, that the famous result of Schoen-Yau, [SY79], and Gromov-Lawson, [GL80a], that there does not exist a metric of positive scalar curvature on any torus extends to the case of foliations. An immediate corollary of our result is the proof of this provided the foliation is spin and has Hausdorff homotopy groupoid.*

**Example 1.9.** *Examples of enlargeable (non-spin) oriented manifolds with  $\hat{A}(M) = 0$  and spin foliations (so accessible by Theorem 1.7, but not by Theorem 1.2 nor Theorem 1.3) with Hausdorff homotopy groupoids are given by  $M = \mathbb{T}^k \times (\mathbb{T}^{6\ell} \# (\mathbb{S}^2 \times \mathbb{C}P^2))$ ,  $k \neq 8\ell + 3, 8\ell + 4$ . See Section 5 for details.*

**Remarks 1.10.** *In theorem 1.7:*

- *The condition on the homotopy groupoid can be changed to requiring that there is a covering of  $M$  so that the induced foliation has Hausdorff holonomy groupoid. Note that a foliation has Hausdorff homotopy groupoid if and only if any covering foliation also does. If its holonomy groupoid is Hausdorff, so too is the holonomy groupoid of any covering foliation, but the converse is false. See [CH97].*
- *Note that there are no dimension restrictions. Note also that the homotopy and holonomy groupoids of any Riemannian foliation are Hausdorff. In addition, there are many important non-Riemannian foliations which also have this property, e.g. those in [H78], [KT79], and [LP76].*

- *Just as in the Gromov-Lawson results, the spin assumption can be weakened to requiring that the foliation induced on some covering of  $M$  has a spin structure.*
- *The condition that  $M$  be enlargeable can be replaced by the condition that  $M$  has an almost flat  $K$ -theory class  $[E]$  with  $\text{ch}([E]) = \text{ch}_0([E]) + \text{ch}_n([E])$  and  $\int_M \text{ch}_n([E]) \neq 0$ .*
- *The requirement that  $f : M \rightarrow \mathbb{S}^n(1)$  be  $\epsilon$  contracting can be loosened to only require that it be  $\epsilon$  contracting on two forms. See [LM89].*

Our approach combines the techniques of Gromov-Lawson, [GL80a, GL80b, GL83] with those of [H95, HL99]. In particular, to prove Theorem 1.7, we need three things: (1) a way to construct a Hermitian bundle on some covering of  $M$  whose curvature is as small as we like and whose Chern character is non-trivial only in dimension  $n$ ; (2) an index theory for elliptic operators defined along the leaves of a foliation which satisfies: (2a) the index of the operator is the same as the “graded dimension” of its kernel; (2b) there is a formula for the index involving characteristic classes. The first requirement is satisfied by assuming that  $M$  is enlargeable. The second and third are provided by the results in [H95, HL99], which are quickly reviewed in the next section.

The main point is that the results of [H95, HL99] remain valid on coverings of compact manifolds provided the leafwise spectrum of the relevant leafwise Dirac operator  $D$  is nice, and  $F$  has Hausdorff homotopy groupoid. If the foliation has uniformly positive scalar curvature, then the spectrum is very nice, in fact there is a gap about zero. This implies that the Chern character of the index bundle must be zero. But this Chern character is the integral over the leaves of  $F$  of the usual expression involving characteristic classes. Then the assumption that  $M$  is enlargeable quickly leads to a contradiction, hence the non-existence of metrics of positive scalar curvature on  $F$ .

For an excellent exposition of the circle of ideas so briefly mentioned here as well as a more nearly complete history of the many contributions of others to this important area, see [LM89].

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## 2. PRELIMINARIES

In this section we briefly recall the results of [H95, HL99] we will use. In particular,  $M$  is a smooth compact oriented manifold of dimension  $n$ , and  $F$  is an oriented foliation of  $M$  of dimension  $p$  with Hausdorff homotopy groupoid. The tangent bundle of  $F$  will be denoted  $TF$ , and we assume that  $TF$  admits a spin structure. We may assume that  $F$  is even dimensional, for if not, we just replace  $M$  by  $M \times \mathbb{S}^1$  and  $F$  by  $F \times \mathbb{S}^1$ . We also assume that  $F$  is spin.

The homotopy groupoid  $\mathcal{G}$  of  $F$  consists of equivalence classes of paths  $\gamma : [0, 1] \rightarrow M$  such that the image of  $\gamma$  is contained in a leaf of  $F$ . Two such paths are equivalent if they are homotopic in their leaf rel their end points. If  $\mathcal{G}$  is Hausdorff, then the foliation induced by  $F$  on any covering of  $M$  is also Hausdorff, [CH97]. This is essential for the results of [H95, HL99] to hold.

There are two natural maps  $r, s : \mathcal{G} \rightarrow M$ , namely  $r([\gamma]) = \gamma(1)$  and  $s([\gamma]) = \gamma(0)$ . The fibers,  $s^{-1}(x)$  for  $x \in M$ , define the foliation  $F_s$  of  $\mathcal{G}$ . Note that  $r : s^{-1}(x) \rightarrow M$  is the simply connected covering of the leaf of  $F$  through  $x$ .

The (reduced) Haefliger cohomology of  $F$ , [Ha80], is given as follows. Let  $\mathcal{U}$  be a finite good cover of  $M$  by foliation charts as defined in [HL90]. For each  $U_i \in \mathcal{U}$ , let  $T_i \subset U_i$  be a transversal and set  $T = \bigcup T_i$ . We may assume that the closures of the  $T_i$  are disjoint. Let  $\mathcal{H}$  be the holonomy pseudogroup induced by  $F$  on  $T$ . Give the space of  $k$ -forms on  $T$  with compact support  $\mathcal{A}_c^k(T)$  the usual  $C^\infty$  topology, and denote the exterior derivative by  $d_T : \mathcal{A}_c^k(T) \rightarrow \mathcal{A}_c^{k+1}(T)$ . Denote by  $\mathcal{A}_c^k(M/F)$  quotient of  $\mathcal{A}_c^k(T)$  by the closure of the vector subspace generated by elements of the form  $\alpha - h^*\alpha$  where  $h \in \mathcal{H}$  and  $\alpha \in \mathcal{A}_c^k(T)$  has support contained in the range of  $h$ . The exterior derivative  $d_T$  induces a continuous differential  $d_H : \mathcal{A}_c^k(M/F) \rightarrow \mathcal{A}_c^{k+1}(M/F)$ . Note that  $\mathcal{A}_c^k(M/F)$  and  $d_H$  are independent of the choice of cover  $\mathcal{U}$ . The associated cohomology theory is denoted  $H_c^*(M/F)$  and is called the Haefliger cohomology of  $F$ . This definition will be extended to non-compact coverings of  $M$  in Section 4.

Denote by  $\mathcal{A}^{p+k}(M)$  the space of smooth  $k$ -forms on  $M$ . As the bundle  $TF$  is oriented, there is a continuous open surjective linear map, called integration over the leaves,

$$\int_F : \mathcal{A}^{p+k}(M) \longrightarrow \mathcal{A}_c^k(M/F)$$

which commutes with the exterior derivatives  $d_M$  and  $d_H$ , so it induces the map

$$\int_F : H^{p+k}(M; \mathbb{R}) \rightarrow H_c^k(M/F).$$

Suppose that  $E$  is a Hermitian vector bundle over  $M$  with Hermitian connection. Then any associated generalized Dirac operator defined along the leaves of  $F$  may be lifted, using the projection  $r : \mathcal{G} \rightarrow M$ , to a generalized Dirac operator  $\not{D}_{E_r}$  along the leaves of the foliation  $F_s$  of  $\mathcal{G}$  with coefficients in the bundle  $E_r = r^*(E)$ . In [H95], a Chern character for  $\not{D}_{E_r}$ , denoted  $\text{ch}(\not{D}_{E_r}^+) \in H_c^*(M/F)$ , is constructed using the Bismut superconnection for foliations. The main result of [H95] is that

$$\text{ch}(\not{D}_{E_r}^+) = \int_F \widehat{A}(TF) \text{ch}(E) \quad \text{in } H_c^*(M/F).$$

Denote by  $P_0^{E_r}$  the graded projection onto the leafwise kernel of  $\not{D}_{E_r}^2$ , which also determines a Chern character  $\text{ch}(P_0^{E_r}) \in H_c^*(M/F)$ . One might hope that, as for compact manifolds,  $\text{ch}(\not{D}_{E_r}^+) = \text{ch}(P_0^{E_r})$ . However, this is not true in general. See [BHW14]. The main result of [HL99] is that if the spectrum of  $\not{D}_{E_r}^2$  is sufficiently well behaved near 0, then

$$\text{ch}(\not{D}_{E_r}^+) = \text{ch}(P_0^{E_r}) \quad \text{in } H_c^*(M/F),$$

so also

$$\int_F \widehat{A}(TF) \text{ch}(E) = \text{ch}(P_0^{E_r}) \quad \text{in } H_c^*(M/F).$$

A special case is when there is a gap about 0 in the spectrum, and in that case we have

$$\int_F \widehat{A}(TF) \text{ch}(E) = \text{ch}(P_0^{E_r}) = 0 \quad \text{in } H_c^*(M/F).$$

### 3. PROOF OF THEOREM 1.7: THE COMPACTLY ENLARGEABLE CASE

Recall definitions 1.4 and 1.5 from the introduction. A manifold is compactly enlargeable if for each  $\epsilon > 0$ , the covering space can be chosen to be compact. For simplicity, we do the compactly enlargeable case first, as the general case requires several technical adjustments for its proof.

Denote by  $\mathcal{S}$  the spin bundle associated to the spin structure on  $TF$ . Then  $\mathcal{S}_E = \mathcal{S} \otimes E$  is a Dirac bundle when restricted to each leaf  $L$  of  $F$ . Define the canonical section  $\mathcal{R}_F^E$  of  $\text{Hom}(\mathcal{S}_E, \mathcal{S}_E)$  by the formula

$$\mathcal{R}_F^E(\sigma \otimes \phi) = \frac{1}{2} \sum_{j,k=1}^p (e_j \cdot e_k \cdot \sigma) \otimes R_{e_j, e_k}^E(\phi),$$

where  $e_1, \dots, e_p$  is an orthonormal local framing of the tangent bundle of  $L$ ,  $R^E$  is the curvature transformation of  $E$ , and the dot is Clifford multiplication.

The following is immediate. For the proof see [LM89], IV.5.

**Lemma 3.1.** *Suppose that  $F$  is a spin foliation of a compactly enlargeable Riemannian  $n$ -dimensional manifold  $M$ . Then, for each  $\epsilon > 0$ , there is a compact orientable Riemannian covering  $\widehat{M} \rightarrow M$  and a Hermitian bundle  $\widehat{E} \rightarrow \widehat{M}$  so that,*

- $\text{ch}(\widehat{E}) = \dim \widehat{E} + \text{ch}_n(\widehat{E})$ ;
- $\int_{\widehat{M}} \text{ch}_n(\widehat{E}) \neq 0$ ;
- $\|\mathcal{R}_{\widehat{F}}^{\widehat{E}}\| \leq \epsilon$ .

where  $\widehat{F}$  is the foliation induced on  $\widehat{M}$  by  $F$ , and  $\mathcal{R}_{\widehat{F}}^{\widehat{E}}$  is the leafwise operator given above.

The following lemma allows us to utilize the results of [HL99].

**Lemma 3.2.** *Suppose that  $F$  is spin foliation of a compactly enlargeable manifold  $M$  which admits a metric whose restriction to the leaves of  $F$  has positive scalar curvature. Let  $(\widehat{M}, \widehat{F})$  and  $\widehat{E}$  be as in Lemma 3.1, and denote the homotopy groupoid of  $\widehat{F}$  by  $\widehat{\mathcal{G}}$ . Then there is a gap about 0 in the spectrum of  $\partial_{\widehat{E}}^2$ .*

*Proof.* Suppose not and let  $\phi \in \text{Im}(P_{\delta}^{E_r})$ , the spectral projection for the interval  $[0, \delta]$ , with  $L^2$  norm equal to 1. We may assume that  $\delta$  is as small as we please. Denote the leafwise scalar curvature for  $\widehat{F}_s$  by  $\kappa$ . Now the leafwise operator  $\partial_{\widehat{E}}^2$  on  $\widehat{\mathcal{G}}$  satisfies the pointwise equality, see [LM89], II.8.17,

$$\partial_{\widehat{E}}^2 = \nabla^* \nabla + \frac{1}{4} \kappa + \mathcal{R}_{\widehat{F}_s}^{\widehat{E}_r}.$$

Choose  $\epsilon$  so small that  $\frac{1}{4} \kappa + \|\mathcal{R}_{\widehat{F}_s}^{\widehat{E}_r}\| > 0$  on  $\widehat{\mathcal{G}}$ . Choose  $c > 0$  so that  $cI \leq \frac{1}{4} \kappa + \mathcal{R}_{\widehat{F}_s}^{\widehat{E}_r}$  on  $\widehat{\mathcal{G}}$ . Then on each leaf  $s^{-1}(\widehat{x})$ ,  $\widehat{x} \in \widehat{M}$ , we have for all small positive  $\delta$ ,

$$\begin{aligned} \delta &\geq \|\partial_{\widehat{E}_r}^2(\phi)\| = \|\partial_{\widehat{E}_r}^2(\phi)\| \|\phi\| \geq \langle \partial_{\widehat{E}_r}^2(\phi), \phi \rangle = \int_{s^{-1}(\widehat{x})} \langle \nabla^* \nabla \phi, \phi \rangle + \int_{s^{-1}(\widehat{x})} \langle (\frac{1}{4} \kappa + \mathcal{R}_{\widehat{F}_s}^{\widehat{E}_r} \phi), \phi \rangle = \\ &\int_{s^{-1}(\widehat{x})} \|\nabla \phi\|^2 + \int_{s^{-1}(\widehat{x})} \langle (\frac{1}{4} \kappa + \mathcal{R}_{\widehat{F}_s}^{\widehat{E}_r} \phi), \phi \rangle \geq \int_{s^{-1}(\widehat{x})} \|\nabla \phi\|^2 + c \|\phi\|^2 \geq c, \end{aligned}$$

an obvious contradiction.  $\square$

**Remark 3.3.** *Note that the same proof shows that there is a spectral gap for  $\partial^2$  on  $\widehat{\mathcal{G}}$ . For this result, we do not need  $M$  to be compactly enlargeable. This same remark holds for the operator  $\partial^2$  on  $\mathcal{G}$ .*

**Proof of Theorem 1.7: the compactly enlargeable case.** The proof is by contradiction. So assume that  $M$  admits a metric whose restriction to the leaves of  $F$  has positive scalar curvature. By Lemma 3.2, both  $\partial^2$  and  $\partial_{\widehat{E}_r}^2$  have spectral gaps about 0. By the results quoted in Section 2, we have

$$\int_{\widehat{F}} \widehat{A}(T\widehat{F}) \text{ch}(\widehat{E}) = \text{ch}(\partial_{\widehat{E}_r}^+) = \text{ch}(P_0^{\widehat{E}_r}) = 0,$$

and

$$\int_{\widehat{F}} \widehat{A}(T\widehat{F}) = \text{ch}(\partial^+) = \text{ch}(P_0) = 0,$$

in the Haefliger cohomology of  $\widehat{F}$ . So for any closed Haefliger current  $C$  associated to  $\widehat{F}$  we have

$$0 = \langle C, \int_{\widehat{F}} \widehat{A}(T\widehat{F}) \text{ch}(\widehat{E}) \rangle = (\dim \widehat{E}) \langle C, \int_{\widehat{F}} \widehat{A}(T\widehat{F}) \rangle + \langle C, \int_{\widehat{F}} \text{ch}_n(\widehat{E}) \rangle.$$

As noted above,  $\int_{\widehat{F}} \widehat{A}(T\widehat{F}) = 0$ , so the first term disappears. If we choose  $C$  to be integration over a complete transversal, then the second term is  $\int_{\widehat{M}} \text{ch}_n(\widehat{E})$ , which is non-zero, a contradiction.

#### 4. PROOF OF THEOREM 1.7: THE GENERAL CASE

For the general case, we assume that the Riemannian cover  $\pi : \widehat{M} \rightarrow M$  is non-compact, and we adapt the proof of Gromov-Lawson as given in [LM89], Section IV.6. This involves extending the results of [H95, HL99] to a non-compact covering space of  $M$ .

For the paper [H95], the main change is in the definition of the Haefliger cohomology for the foliation  $\widehat{F}$  of  $\widehat{M}$ . The (reduced) Haefliger cohomology of  $\widehat{F}$  is given as follows. The good cover  $\mathcal{U}$  of  $M$  determines a good cover  $\widehat{\mathcal{U}}$  of  $\widehat{M}$  which consists of all open sets  $\widehat{U}_{i,j}$  so that  $\pi : \widehat{U}_{i,j} \rightarrow U_i$  is a diffeomorphism, where  $U_i \in \mathcal{U}$ . That is  $\cup_j \widehat{U}_{i,j} = \pi^{-1}(U_i)$ . The transversal  $T_i \subset U_i$  determines the transversal  $\widehat{T}_{i,j} \subset \widehat{U}_{i,j}$ , and the closures

of the  $\widehat{T}_{i,j}$  are disjoint. The space  $\mathcal{A}_c^k(\widehat{T})$  consists of all smooth  $k$ -forms on  $\cup_{i,j} \widehat{T}_{i,j}$  which have compact support in each  $\widehat{T}_{i,j}$ , and such that they are uniformly bounded in the usual  $C^\infty$  topology. As above, we have the exterior derivative by  $d_{\widehat{T}} : \mathcal{A}_c^k(\widehat{T}) \rightarrow \mathcal{A}_c^{k+1}(\widehat{T})$ .

The holonomy pseudogroup  $\widehat{\mathcal{H}}$  acts on  $\mathcal{A}_c^k(\widehat{T})$  just as  $\mathcal{H}$  does on  $\mathcal{A}_c^k(T)$ . Denote by  $\mathcal{A}_c^k(\widehat{M}/\widehat{F})$  the quotient of  $\mathcal{A}_c^k(\widehat{T})$  by the closure of the vector subspace  $\widehat{V}$  generated by elements of the form  $\alpha - h^*\alpha$  where  $h \in \widehat{\mathcal{H}}$  and  $\alpha \in \mathcal{A}_c^k(\widehat{T})$  has support contained in the range of  $h$ . We need to take great care as to what “the vector subspace  $\widehat{V}$  generated by elements of the form  $\alpha - h^*\alpha$ ” means. This is especially important in the proof of Lemma 3.12 of [H95]. Elements of  $\widehat{V}$  consist of possibly infinite sums of elements of the form  $\alpha - h^*\alpha$ , with the following restriction: there is  $n \in \mathbb{N}$  so that the number of elements having the domain of  $h$  contained in any  $\widehat{T}_{i,j}$  is less than  $n$ .

The exterior derivative  $d_{\widehat{T}}$  induces a continuous differential  $d_{\widehat{H}} : \mathcal{A}_c^k(\widehat{M}/\widehat{F}) \rightarrow \mathcal{A}_c^{k+1}(\widehat{M}/\widehat{F})$ . Note that  $\mathcal{A}_c^k(\widehat{M}/\widehat{F})$  and  $d_{\widehat{H}}$  are independent of the choice of cover  $\mathcal{U}$ . The associated cohomology theory is denoted  $H_c^*(\widehat{M}/\widehat{F})$  and is called the Haefliger cohomology of  $\widehat{F}$ .

Let  $f : \widehat{M} \rightarrow \mathbb{S}^n$  be an  $\epsilon$  contracting map which is constant near infinity and has non-zero degree. Because  $f$  is constant near infinity, the pull back of any bundle on  $\mathbb{S}^n$  under  $f$  is a trivial bundle off some compact subset  $K$  of  $\widehat{M}$ . Using the Chern-Weil construction of characteristic classes, we have that the Chern character of such a bundle has compact support contained in  $K$  and it is immediate that Lemma 3.1 extends to  $\widehat{M}$ . Let  $\widehat{E} \rightarrow \widehat{M}$  be the bundle of Lemma 3.1, and denote by  $I^n$  the trivial Hermitian bundle of dimension  $n$  over  $\widehat{M}$ . We may assume that the connection on  $\widehat{E}$  used to construct its characteristic classes is compatible with that on  $I^n$  off  $K$ . Thus the K-theory class  $[\widehat{E}] - [I^n]$  is supported on  $K$ , and its Chern character is  $\text{ch}([\widehat{E}]) - n = \text{ch}_n(\widehat{E})$ .

The arguments in [H95, HL99] are arguments on the homotopy groupoid of  $\widehat{F}$ . The fact that the geometries of  $\widehat{M}$  and  $\widehat{F}$  are uniformly bounded, implies that those results are equally valid here. In particular we have that, in  $H_c^*(\widehat{M}/\widehat{F})$ ,

$$\text{ch}(\partial_{\widehat{E}_r}^+) = \int_{\widehat{F}} \widehat{A}(T\widehat{F}) \text{ch}(\widehat{E}) \quad \text{and} \quad \text{ch}(\partial_{I^n}^+) = \int_{\widehat{F}} \widehat{A}(T\widehat{F}) \text{ch}(I^n).$$

The proof of Lemma 3.2 works equally well here, so there are spectral gaps about 0 for  $\partial_{\widehat{E}_r}^2$  and  $\partial_{I^n}^2$ . Using this fact and applying the results of [H95, HL99], we have

$$\text{ch}(\partial_{\widehat{E}_r}^+) = \text{ch}(P_0^{\widehat{E}_r}) = 0 \quad \text{and} \quad \text{ch}(\partial_{I^n}^+) = \text{ch}(P_0^{I^n}) = 0,$$

in  $H_c^*(\widehat{M}/\widehat{F})$ . So

$$\int_{\widehat{F}} \widehat{A}(T\widehat{F}) \text{ch}(\widehat{E}) = \int_{\widehat{F}} \widehat{A}(T\widehat{F}) \text{ch}(I^n) = 0,$$

and

$$0 = \int_{\widehat{F}} \widehat{A}(T\widehat{F}) \text{ch}(\widehat{E}) - \int_{\widehat{F}} \widehat{A}(T\widehat{F}) \text{ch}(I^n) = \int_{\widehat{F}} \widehat{A}(T\widehat{F}) \text{ch}_n(\widehat{E})$$

in  $H_c^*(\widehat{M}/\widehat{F})$ .

Since we are using Chern-Weil theory to construct differential forms which represent our characteristic classes, we have that  $\widehat{A}(T\widehat{F}) \text{ch}(\widehat{E}) - \widehat{A}(T\widehat{F}) \text{ch}(I^n)$  is represented by a differential form with support in the compact subset  $K$ , so

$$\int_{\widehat{M}} \widehat{A}(T\widehat{F}) \text{ch}(\widehat{E}) - \widehat{A}(T\widehat{F}) \text{ch}(I^n) = \int_{\widehat{M}} \widehat{A}(T\widehat{F}) [\text{ch}(\widehat{E}) - \text{ch}(I^n)]$$

is well defined and it equals

$$\int_{\widehat{M}} \widehat{A}(T\widehat{F}) \text{ch}_n(\widehat{E}) = \int_{\widehat{M}} \text{ch}_n(\widehat{E}) \neq 0$$

by construction.



But just as in the case of a compact covering of  $M$ , we may pair the compactly supported Haefliger class  $\int_{\widehat{F}} \widehat{A}(T\widehat{F}) \text{ch}_n(\widehat{E})$  with the closed Haefliger current  $C$  which is integration over the complete transversal  $\widehat{T}$ , to obtain

$$\int_{\widehat{M}} \widehat{A}(T\widehat{F}) \text{ch}_n(\widehat{E}) = \langle C, \int_{\widehat{F}} \widehat{A}(T\widehat{F}) \text{ch}_n(\widehat{E}) \rangle = \langle C, 0 \rangle = 0,$$

a contradiction.

## 5. FINAL NOTES

Connes has vanishing results for characteristic numbers of foliations with positive scalar curvature. One of which is

**Theorem 5.1.** (*Corollary 8.3 [C86]*) *Suppose that  $F$  is a spin foliation of a compact oriented manifold  $M$ , and that  $F$  admits a metric of positive scalar curvature. Let  $\mathcal{R}$  be the subring of  $H^*(M; \mathbb{R})$  generated by the Chern characters of holonomy equivariant bundles. Then  $\langle \widehat{A}(TF)\omega, [M] \rangle = 0$  for all  $\omega \in \mathcal{R}$ .*

Note that any complex bundle associated to the normal bundle  $\nu$  of  $F$  is holonomy equivariant. The important property for our approach is that the pull back of the bundle to  $F_s$  under  $r : \mathcal{G} \rightarrow M$  be leafwise almost flat. Holonomy equivariant bundles have this property. Using the techniques of this paper, it is then fairly straightforward to extend this result to the vanishing of Haefliger cohomology classes as follows.

**Theorem 5.2.** *Suppose that  $F$  is a spin foliation of a compact oriented manifold  $M$  with Hausdorff homotopy groupoid, and that  $F$  admits a metric of positive scalar curvature. Let  $\mathcal{T}$  be the subring of  $H^*(M; \mathbb{R})$  generated by the Chern characters of bundles whose pull back to  $F_s$  under  $r : \mathcal{G} \rightarrow M$  is leafwise almost flat. Then  $\int_F \widehat{A}(TF)\omega = 0$  in  $H_c^*(M/F)$  for all  $\omega \in \mathcal{T}$ .*

**Corollary 5.3.** *Let  $F$  and  $\mathcal{T}$  be as above. Then for any Haefliger current  $\beta$ ,  $\langle \int_F \widehat{A}(TF)\omega, \beta \rangle = 0$ .*

Taking  $\beta$  to be the Haefliger current given by integration over a complete transversal gives Theorem 5.1 for foliations with Hausdorff homotopy groupoid, with  $\mathcal{R}$  replaced by  $\mathcal{T}$ . In particular, taking  $\omega = \widehat{A}(\nu)$  gives Theorem 1.2 for such foliations.

**Details of Example 1.9.** Recall that  $M = \mathbb{T}^k \times (\mathbb{T}^6 \sharp (\mathbb{S}^2 \times \mathbb{C}P^2))$ . Since  $\mathbb{T}^k$  and  $\mathbb{T}^6$  are enlargeable, so is  $M$ . Since  $\widehat{A}$  is multiplicative and  $\widehat{A}(\mathbb{T}^k) = 0$ ,  $\widehat{A}(M) = 0$ . The manifold  $\mathbb{T}^6 \sharp (\mathbb{S}^2 \times \mathbb{C}P^2)$  is well known to be non-spin (also its universal cover is non spin). Now the Stiefel-Whitney classes of these manifolds satisfy

$$w_1(\mathbb{T}^k) = w_2(\mathbb{T}^k) = w_1(\mathbb{T}^6 \sharp (\mathbb{S}^2 \times \mathbb{C}P^2)) = 0 \quad \text{and} \quad w_2(\mathbb{T}^6 \sharp (\mathbb{S}^2 \times \mathbb{C}P^2)) \neq 0.$$

Denote the obvious projections by  $\pi_1$  and  $\pi_2$ . Then it is clear that  $w_1(M) = 0$ , and

$$\begin{aligned} w_2(M) &= \pi_1^*(w_2(\mathbb{T}^k)) + \pi_1^*(w_1(\mathbb{T}^k)\pi_2^*(w_1(\mathbb{T}^6 \sharp (\mathbb{S}^2 \times \mathbb{C}P^2))) + \pi_2^*(w_2(\mathbb{T}^6 \sharp (\mathbb{S}^2 \times \mathbb{C}P^2))) = \\ &\quad \pi_2^*(w_2(\mathbb{T}^6 \sharp (\mathbb{S}^2 \times \mathbb{C}P^2))) \neq 0. \end{aligned}$$

$M$  admits many interesting spin foliations with Hausdorff homotopy groupoids. In particular,  $\mathbb{T}^k$  admits a large class of spin foliations  $F$  with Hausdorff homotopy groupoids, and these induce foliations on  $M$ , of the same dimension with the same properties, namely  $\pi_1^*(TF)$ .

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